*Strategic Support of Cooperation in **n**-person Differential Games with Prescribed Duration and Dependent Motions*

Speaker **Petrosyan L. A.**

Saint-Petersburg State University,
the faculty of Applied Mathematics and Control Processes

Independent Motions

$$\dot{x}^i = f^i\left(x, u_1, \ldots, u_n\right),$$

$$x^i(t_0) = x_0^i,$$

$$i \in \{1, \ldots, n\} = N; \qquad x^i \in R^k; \qquad u_i \in U_i \subset CompR^l$$

$$t \in [t_0, T]$$

$$K_i\left(x_0, T - t_0; u_1, \ldots, u_n\right) = \int_{t_0}^{T} h_i\left(x(t)\right)dt$$

$$x(t) = \left\{x^1(t), \ldots, x^n(t)\right\}, \qquad x_0 = \left(x_0^1, \ldots, x_0^n\right)$$

$$\Gamma\left(x_0, T - t_0\right)$$

Cooperation

$$V\left(x_0, T-t_0; N\right) = \max_{\substack{u_i \\ i\in N}} \sum_{i=1}^{n} K_i\left(x_0, T-t_0; u_1, \ldots, u_n\right) = \sum_{i=1}^{n} \int_{t_0}^{T} h_i\left(\bar{x}(t)\right) dt$$

$\bar{x}(t)$ – cooperative trajectory $\left(\bar{x}(t) = \{\bar{x}_1(t), \ldots, \bar{x}_n(t)\}\right)$

$V\left(x_0, T-t_0; \{i\}\right)$, $i \in N$ – value functions – value of the zero-sum game between $\{i\}, \{N \setminus i\}$ with payoff $K_i\left(x_0, T-t_0; u_1, \ldots, u_n\right)$

Imputation

$$\xi_i\left(x_0, T-t_0\right)$$

$$\sum_{i\in N}\xi_i\left(x_0, T-t_0\right)=V\left(x_0, T-t_0; N\right),$$

$$\xi_i\left(x_0, T-t_0\right)\geq V\left(x_0, T-t_0; \{i\}\right) \quad i\in N$$

IDP

$$\beta_i(\tau), \quad \tau\in\left[t_0, T\right]$$

$$\xi_i\left(x_0, T-t_0\right)=\int_{t_0}^{T}\beta_i(\tau)d\tau$$

$$\Gamma\left(\bar{x}(t), T-t\right) \quad - \text{Subgame along } \bar{x}(t),\ t \in \left[t_0, T\right]$$

$$V\left(\bar{x}(t), T-t; N\right), \quad V\left(\bar{x}(t), T-t; \{i\}\right), \quad \xi_i\left(\bar{x}(t), T-t\right)$$

$$\sum_{i\in N} \xi_i\left(\bar{x}(t), T-t\right) = V\left(\bar{x}(t), T-t; N\right),$$

$$\xi_i\left(\bar{x}(t), T-t\right) \geq V\left(\bar{x}(t), T-t; \{i\}\right) \qquad i \in N$$

$$\xi_i\left(\bar{x}(t), T-t\right) = \int_t^T \beta_i(\tau)\,d\tau$$

$$\beta_i(t) = -\frac{d}{dt}\xi_i\left(\bar{x}(t), T-t\right)$$ **– *time-consistency condition***

How we supported cooperation in the case of independent motion?

New game $\overline{\Gamma}\left(x_0, T - t_0\right)$

for any $x(t)$ let $\bar{t} = \inf\{t : x(t) \neq \bar{x}(t)\}$,

new payoff: $\overline{K}_i\left(x_0, T - t_0\right) = \int_{t_0}^{\bar{t}} \beta_i(\tau)d\tau + \int_{\bar{t}}^{T} h\left(x(\tau)\right)d\tau$

Punishing the deviator from cooperation

If $\hat{t}$ is the first moment of time when some player $k \in N$ deviates from cooperative trajectory, this will be **fixed** by other players (since they have the information about $x^i(t)$) and after some small time $\varepsilon > 0$ all players (except k) will play against $k \in N$. In this case k can get not more than $V\left(\bar{x}(\hat{t}), T - \hat{t}, \{k\}\right) + \varepsilon$, but if he continues to cooperate he would get $\xi_i\left(\bar{x}(t), T - t\right) \geq V\left(\bar{x}(\hat{t}), T - \hat{t}, \{k\}\right)$.

Thus he will be punished with accuracy $\varepsilon > 0$

But if the motions are dependent this is impossible.
What to do?

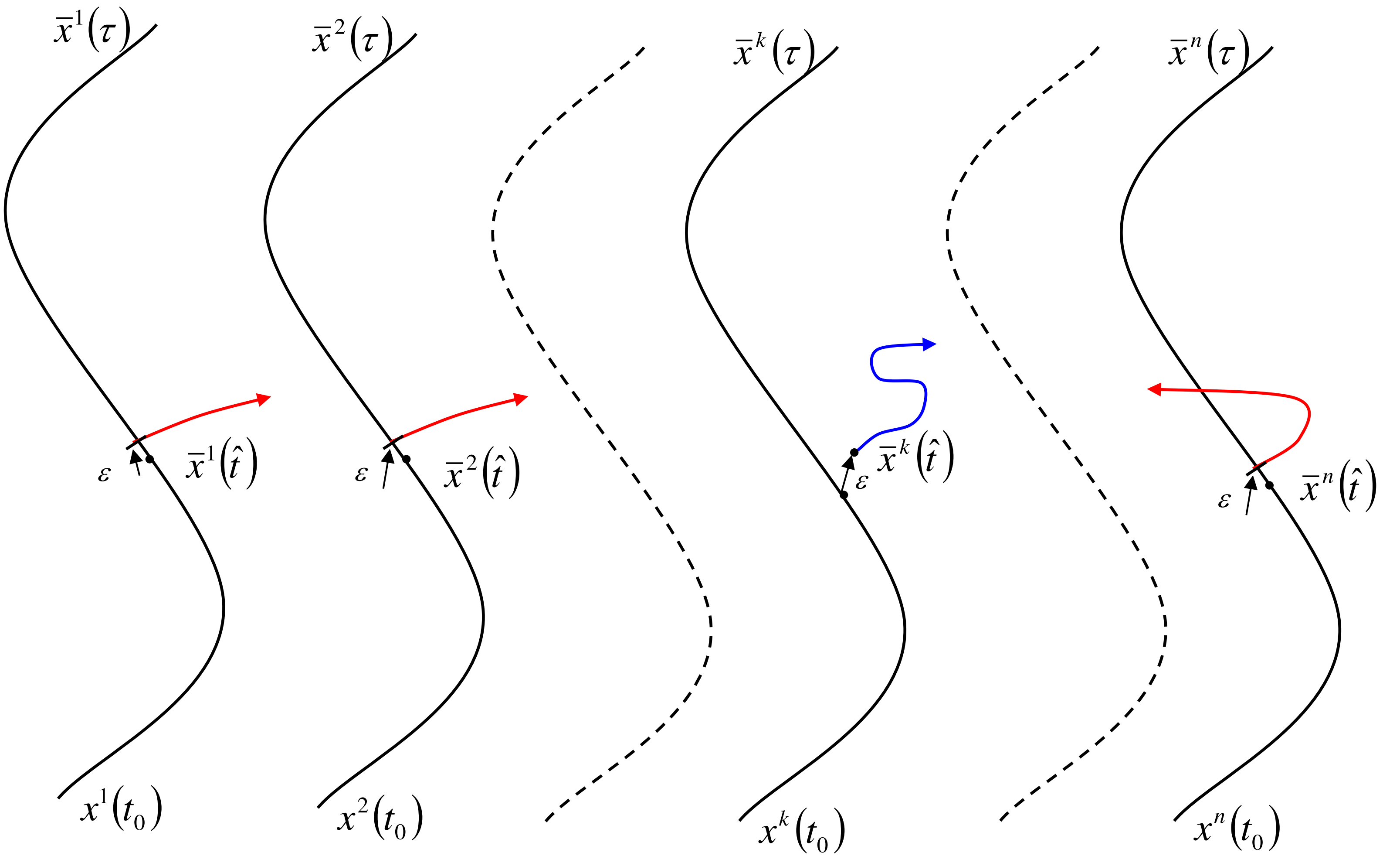
$\bar{x}^1(\tau)$
$\bar{x}^2(\tau)$
$\bar{x}^k(\tau)$
$\bar{x}^n(\tau)$
ε
$\bar{x}^1(\hat{t})$
ε
$\bar{x}^2(\hat{t})$
$\bar{x}^k(\hat{t})$
ε
ε
$\bar{x}^n(\hat{t})$
$x^1(t_0)$
$x^2(t_0)$
$x^k(t_0)$
$x^n(t_0)$

Figure 1

Dependent Motions

$$\dot{x} = f\left(x, u_1, \ldots, u_n\right),$$

$$x(t_0) = x_0,$$

$$i \in \{1, \ldots, n\} = N; \quad x^i \in R^k; \quad u_i \in U_i \subset CompR^l$$

$$t \in \left[t_0, T\right]$$

$$K_i\left(x_0, T - t_0; u_1, \ldots, u_n\right) = \int_{t_0}^{T} h_i\left(x(t)\right) dt$$

$$\Gamma\left(x_0, T - t_0\right)$$

Cooperation

$$V(x_0, T-t_0; N) = \max_{\substack{u_i \\ i \in N}} \sum_{i=1}^{n} K_i(x_0, T-t_0; u_1, \ldots, u_n) = \sum_{i=1}^{n} \int_{t_0}^{T} h_i(\bar{x}(t))dt$$

$\bar{x}(t)$ – cooperative trajectory

$\tilde{V}(x_0, T-t_0; \{i\})$, $i \in N$ – new value functions - payoff of player i in a fixed subgame perfect NE, $(\tilde{u}_1, \ldots, \tilde{u}_n) = \tilde{u}$

The definition of imputation as previous only we change $V(x_0, T-t_0; \{i\})$ on $\tilde{V}(x_0, T-t_0; \{i\})$, $i \in N$

Everything what follows – the same.
Only the punishment is different.

If one of players, say player $k \in N$ deviates first at moment $\hat{t}$ then after $\varepsilon > 0$ (ε small) not deviating player l will observe the deviation, but he has only x, so he doesn't know who deviated. What to do?

Switch to the strategy from subgame perfect NE $\tilde{u}_l$

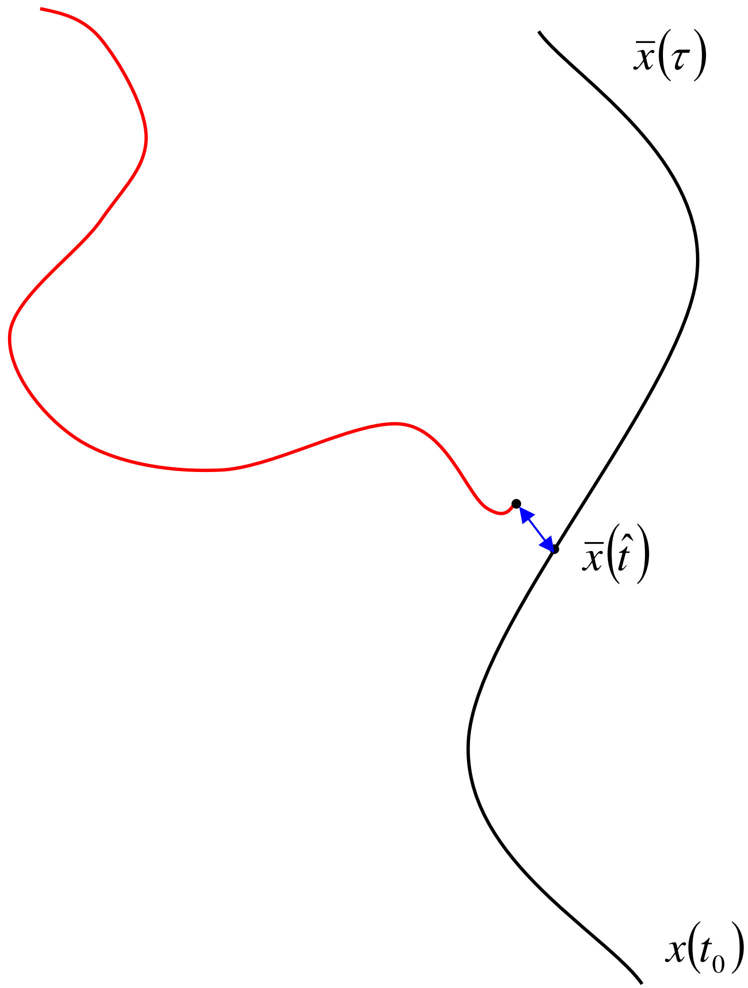

Figure 2

Results

As result the deviator will be punished with ε - accuracy, since

$$\int_{\hat{t}}^{T} \beta_k(\tau) d\tau \geq \tilde{V}\left(\hat{x}(\hat{t}), T - \hat{t}; \{k\}\right) - \varepsilon$$

for all $i \in N$, including the deviator, which may remain unknown

References

1. Petrosjan L.A., Zenkevich N.A. Principles of stable cooperation, The mathematical games theory and its applications, 1, 2009, pp.102-117 (in Russian)

2. Chistyakov S.V. To the solution of game problem of pursuit, Prikl. Math. i Mech., 5, 1977, 825-832, (in Russian).

3. Chistyakov S.V. Operatory znacheniya antagonisticheskikx igr (Value Operators in Two-Person Zero-Sum Differential Games), St. Petersburg: St. Petersburg Univ. Press, 1999.

4. Chentsov A.G. On a game problem of convering at a given instant time, Math. USSR Sbornic, 3 1976, 353-376.

5. Chistyakov S.V. O beskoalizionnikx differenzial'nikx igrakx (On Coalition-Free Differential Games) Dokl. Akad. Nauk. 1981. Vol. 259, no. 5, pp. 1052-1055; English transl. in Soviet Math. Dokl. 24, 1981, no. 1, pp.166-169.

6. Petrosjan L.A. Differential Games of Pursuit. World Scientific, Singapore, 1993

7. Pschenichny B.N. E-strategies in differential games//Topics in Differential Games. New York, London, Amsterdam. 1973, pp. 45-56

8. Fridman A. Differential Games // John Wiley and Sons, New York, NY, 1971.

9. Petrosjan L.A., Danilov N.N. Stability of Solutions in nonzero-sum Differential Games with Integral Payoffs, Viestnik Leningrad University, 1979, N1, pp.52-59.

10. Petrosjan L.A. The Shapley Value for Differential Games, Annals of the International Society of Dynamic Games, Vol.3, Geert Jan Olsder Editor, Birkhauser, 1995, pp. 409-417.

Thanks!

Any questions?